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STRUCTURE OF TENSOR FIELDS OF THE SECOND RANK GENERATED BY AN INCOMPATIBILITY OPERATOR*

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There is considered the structure of fields of symmetric tensors $\{\eta\}$ generated by the incompatibility operator Ink acting according to the formula $\text{Ink}\,\eta \equiv \nabla \times \eta \times \nabla$. This operator which is closely related to internal stresses in bodies was considered by Kroner /1/ in application to the continuum theory of dislocations. Allied to the study of the structure of tensor fields is the problem of their restoration according to a given incompatibility and divergence as well as the decomposition into incompatible and compatible strain. For a sufficiently smooth Kröner tensor field that vanishes at infinity, by starting from the analogy with the properties of a vector field, it is shown that such a decomposition exists and is unique. In the supplement to /2/, this problem is solved in practice, where an effective algorithm is developed for the decomposition into appropriate invariant components by using projection operators. Analogous questions are examined below in application to finite domains as well as in connection with the decomposition of a tensor into deviatoric and spherical parts.

1. Let a field of a symmetric tensor η, called compatible if $\text{Ink}\,\eta = 0$, be given in a domain V with boundary Γ; all the remaining tensors are incompatible.

Definition 1.1. We call an incompatible tensor $\mathbf{s}$ in incompatibility tensor if it is a solution of the system

$$\text{Ink}\,\mathbf{s}(\mathbf{x}) = \boldsymbol{\varkappa}(\mathbf{x}),\quad \text{div}\,\mathbf{s}(\mathbf{x}) = 0,\quad \mathbf{x} \in V \tag{1.1}$$

which becomes identically zero together with the tensor $\boldsymbol{\varkappa}$.

By solving this system, the incompatibility tensor $\mathbf{s}$ is restored by its image $\boldsymbol{\varkappa}$ obtained by using the operator Ink; any (symmetric) tensor with zero divergence can emerge as the image. The operator Res, inverse to Ink is thereby determined by tensors with zero divergence. The operators Res and Ink are mutually reciprocal in the set of such tensors.

The Definition 1.1 introduced differs from the traditional definition /1 – 4/ when representability in the form $\mathbf{s} = \text{Ink}\,\eta$ is the basis for the classification. Both definitions are equivalent in tensor fields given in all space and regular at infinity. However, in the case of a finite domain fields of compatible tensors with zero divergence exist which should be referred to incompatibility tensor fields according to the traditional definition. Definition 1.1 eliminates this incorrectness.

We introduce the following notation

$$r \equiv |\mathbf{x} - \mathbf{y}|,\quad \Pi_V(r\eta) \equiv -\frac{1}{8\pi}\iiint\limits_V r\eta(\mathbf{y})\,dV_y,\quad \rho \equiv \frac{2}{|\mathbf{x} - \mathbf{y}|}$$

where $\mathbf{x}, \mathbf{y}$ are radius-vectors of the running and integration points. We correspondingly denote the Newtonian potential by $\Pi_V(\rho\eta)$. In the case of integration over an infinite domain, we replace the subscript V by ∞. We denote the external normal to the boundary Γ by $\mathbf{n}$ and the Laplace operator by Δ.

Let $\text{div}\,\eta = 0$, and we examine $\text{div}\,\Pi_V$. If the tensor η or its external vector $\mathbf{n} \cdot \eta$ vanishes on Γ, then $\text{div}\,\Pi_V = 0$; this is not so in the general case.

We obtain an expression analogous to Π_V and possessing zero divergence. To do this, we find the deformation tensor $\mathbf{q}$, possessing zero divergence, outside the volume V by means of the external vector $\chi \equiv \mathbf{n} \cdot \eta$ given on Γ, and we continue η in all space to infinity in a regular manner, i.e., we take the solution $\mathbf{q}$ of the following problem for $\mathbf{x} \notin V$

$$\text{Ink}\,\mathbf{q}(\mathbf{x}) = 0,\quad \text{div}\,\mathbf{q}(\mathbf{x}) = 0;\quad \mathbf{n} \cdot \mathbf{q}(\mathbf{x})_\Gamma = \chi(\mathbf{x}),\ \mathbf{x} \in \Gamma \tag{1.2}$$

as the continuation of η_e.

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This problem reduces to a problem in the vector field $\mathbf{u}(\mathbf{x})$

$$\Delta\mathbf{u} + \text{grad div}\,\mathbf{u} = 0, \quad \frac{\partial\mathbf{u}}{\partial\mathbf{n}} + \frac{1}{2}\mathbf{n}\times\text{rot}\,\mathbf{u}|_\Gamma = \chi \qquad (1.3)$$

which agrees with the second boundary value problem of elasticity theory for zero Poisson's ratio. As is known /4/, the solution of the external problem (1.3) exists, is unique, and is regular at infinity. Hence the continued tensor

$$\eta_\infty = \begin{cases} \eta, & \mathbf{x}\in V \\ \eta_e, & \mathbf{x}\notin V \end{cases}$$

will satisfy the condition $\text{div}\,\eta_\infty = 0$ in all space, will diminish at the rate $||\mathbf{x}|^2\eta_\infty(\mathbf{x})| < M = \text{const}$ at infinity, and its external vector on Γ will not undergo discontinuities (the theorem on divergence can therefore be applied). The potential $\Pi_\infty(r\eta_\infty)$ constructred on its basis has zero divergence; its contraction in the domain V

$$\Pi_\infty^V(r\eta_\infty) = \Pi_\infty(r\eta_\infty), \quad \mathbf{x}\in V \qquad (1.4)$$

will also possess this property.

Lemma 1.1. A regular solution at infinity for the external problem (1.2) is a diviator if the boundary condition χ is the external deviator vector.

Proof. Because of the first equation (1.2) there exists a vector $\mathbf{u}(\mathbf{x})$ such that $\mathbf{q}(\mathbf{x}) = \text{def}\,\mathbf{u}(\mathbf{x})$, where def is the operator for the formulation of the deformation tensor in the displacement vector. The result known in elasticity theory of the harmonicity of the spherical part of the deformation tensor follows from (1.3). Let us formulate the external boundary value problem for it that corresponds to the problem under consideration: at infinity the spherical tensor should be regular and should correspond to the boundary condition χ on Γ. We represent the vector χ in the form of a sum of tangential χ_τ and χ_n normal components to Γ. It is easy to see that χ_τ and χ_n depend, respectively, only on the deviator and the spherical part. If $\chi_n(\mathbf{x}) = 0$ then we obtain a harmonic problem with zero value on the boundary and regularity condition at infinity for the spherical tensor. Such a harmonic tensor is evidently identical to zero.

2. We show that the field of the tensor η is restored uniquely within the finite domain V as a solution of the problem

$$\text{Ink}\,\eta(\mathbf{x}) = \varkappa(\mathbf{x}), \quad \text{div}\,\eta(\mathbf{x}) = \mathbf{v}(\mathbf{x}), \quad \mathbf{x}\in V; \quad \mathbf{n}\cdot\eta|_\Gamma = \chi(\mathbf{x}), \quad \mathbf{x}\in\Gamma \qquad (2.1)$$

We shall seek the general solution in the form of the sum $\mathbf{s} + \xi + \mathbf{q}$ of particular solutions of the first two equations and the general solution of the homogeneous system for the appropriate boundary condition.

We find the tensor $\mathbf{s}$ as a particular solution of the problem (1.1). To do this we use the identity /2/

$$\Delta^2 = \text{def}\,(2\Delta - \text{grad div})\,\text{div} + \text{Ink Ink} \qquad (2.2)$$

Existence of a tensor $\mathbf{h}$ such that $\mathbf{s} = \text{Ink}\,\mathbf{h}$ follows from the second equation in (1.1). Then it is satisfied indentically, and if we set $\text{div}\,\mathbf{h} = 0$, then by using (2.2) we can conclude that the first equation in (1.1) takes the form $\Delta^2\mathbf{h} = \varkappa$. Its particular solution that satisfies the condition $\text{div}\,\mathbf{h} = 0$ is a potential of the type (1.4), say. We then obtain the solution of the problem (1.1) in the form

$$\mathbf{s}(\mathbf{x}) = \text{Ink}\,\Pi_\infty^V(r\varkappa_\infty) \qquad (2.3)$$

The solution (2.3) is obtained by analogy with finding the vortex component of the vector field /5/ when the particular solution of the system $\text{rot}\,\mathbf{u}(\mathbf{x}) = \mathbf{v}(\mathbf{x}),\ \text{div}\,\mathbf{u}(\mathbf{x}) = 0,\ \mathbf{x}\in V$, similar to the problem (1.1), is sought by using a vector analysis identity, $\Delta = \text{grad div} - \text{rot rot}$ analogous to (2.2) by assuming $\mathbf{u} = \text{rot}\,\mathbf{w}$ and $\text{div}\,\mathbf{w} = 0$.

It should be noted that the method recommended in /6/ that does not require satisfaction of the condition $\text{div}\,\mathbf{w} = 0$, does not permit solution of the problem in the general case. Thus if the solution is selected in the recommended form

$$\mathbf{w} = -\Pi_V(\rho\mathbf{v}) + \text{grad}\,\psi, \quad \Delta\psi = \text{div}\,\Pi_V(\rho\mathbf{v})$$

then because $\Delta\mathbf{w} = -\mathbf{v}$ the equality $\Delta\,\text{grad}\,\psi = 0$ should be satisfied, which will result in a condition on the vector $\mathbf{v}$

$$\text{grad div}\,\Pi_V(\rho\mathbf{v}) = 0$$

This is related to the fact that not every harmonic vector must be a potential.

We find the tensor ξ as a particular solution of the problem

$$\operatorname{Ink} \xi(\mathbf{x}) = 0, \ \operatorname{div} \xi(\mathbf{x}) = \mathbf{v}(\mathbf{x}), \ \mathbf{x} \in V \tag{2.4}$$

To do this, we decompose the vector $\mathbf{v}$ into vortex $\mathbf{v}_r$ and potential $\mathbf{v}_g$ parts s /5/, and we seek ξ in the form of the corresponding sum $\xi = \xi_r + \xi_g$. Because of the linearity of the problem, (2.4) decomposed into two. We set $\xi_r = 2 \operatorname{def} \mathbf{u}_r$, where $\mathbf{u}_r = \operatorname{rot} \mathbf{z}$ and $\mathbf{v}_r = \operatorname{rot} \mathbf{w}$. This results in a particular solution in the form $\xi_r = 2 \operatorname{def} (\operatorname{rot} \mathbf{z})$, $\mathbf{z} = \Pi_V (\rho \mathbf{w})$. Correspondingly, we set $\xi_g = \operatorname{def} \mathbf{u}_g$, where $\mathbf{u}_g = \nabla\varphi$ and $\mathbf{v}_g = \nabla\psi$. This results in a particular solution of the form $\xi_g = \nabla\nabla\varphi$, $\varphi = \Pi_V (\rho\psi)$.

Let us introduce the notation: $\mathbf{n} \cdot \mathrm{s} |_\Gamma \equiv \chi_1(\mathbf{x})$, $\mathbf{n} \cdot \xi |_\Gamma \equiv \chi_2(\mathbf{x})$, $\chi_3(\mathbf{x}) \equiv \chi(\mathbf{x}) - \chi_1(\mathbf{x}) - \chi_2(\mathbf{x})$. By direct substitution it can be seen that the sum $\mathrm{s} + \xi$ will satisfy the equations in (2.1) and there remains to find $\mathbf{q}$ as a solution of the internal problem (1.2) under the boundary condition $\mathbf{n} \cdot \mathbf{q}(\mathbf{x}) |_\Gamma = \chi_3(\mathbf{x})$. This latter reduces to (1.3), where the second equation in (1.2) assures the existence of the solution /4/, which is unique for the tensor $\mathbf{q}$.

Therefore, the solution of the problem (2.1) is obtained in the form

$$\eta = \mathrm{s} + \xi + \mathbf{q} \tag{2.5}$$

where s is the incompatibility tensor, and ξ and $\mathbf{q}$ are strains. The tensor $\mathbf{q}$ which is compatible, can be represented in the form $\mathbf{q} = \operatorname{Ink} \mathbf{h}$, which would result in the introduction of the Definition 1.1.

If the required tensor $\eta \in C^{(2)}(V)$. then also the tensors $\mathrm{s}, \xi \in C^{(2)}(V)$ since they are obtained by differentiating potentials with densities of appropriate smoothness. Hence s and ξ are single-valued functions and only the tensor $\mathbf{q}$ can be the source of the multivaluedness (for a multiconnected surface domain V).

The representation (2.5) simultaneously solves the problem of decomposing the arbitrary tensor η into the incompatibility tensor s and the compatible tensor $\xi' = \xi + \mathbf{q}$, by also verifying the validity of the Kröner decomposition $\eta = \mathrm{s} + \xi'$ /1/ for finite domains. To do this it is sufficient to take s from (2.3) and to define ξ' as the difference $\eta - \mathrm{s}$.

3. The operator Ink is a second-order matrix linear differential operator, and it is consequently evident that there exist among the tensors those that annihilate them because of single or repeated multiple application. Tensors also exist which do not annihilate them at all /2/.

We call the tensor $\eta(p)$ a nilpotent element of the operator Ink if a natural number p exists such that $\operatorname{Ink}^p \eta(p) = 0$. We call the least natural number the height of the element $\eta(p)$.

Compatible tensors have a height one, while the incompatible are greater than one. For convenience, we do not ascribe the index $p = \infty$ to nilpotent elements. The operator Ink lowers while the operator Res raises the height of a nilpotent element by one.

We call the set of all nilpotent elements of height p the nilpotency class $\boldsymbol{p}$. The possibility of decomposition by class is eivdent since the sets of elements of different height do not intersect by pairs. Using tensors with components from polynomials of finite degree, the example can be presented of a tensor of any previously assigned height. By comparing the class with its height, all the classes can be numbered. Therefore, the operator Ink generates the decomposition of a linear space of tensor fields into a countable set of nilpotency classes.

The set of nilpotent tensors is a subspace of the linear tensor space. The nilpotency class 1 is the kernel of the operator Ink, and therefore, is also a subspace. The remaining classes are not subspaces since they do not contain the zero element.

We examine subgroups of an additive group of linear spaces of tensor fields. If necessary, we predefine the appropriate sets by a zero tensor so that they would form a subgroup. Let H, S and K be subgroups of incompatible tensors η, the tensors of incompatibility s, and the tensors $\varkappa$ with zero divergence. Furthermore, let $\mathrm{H}(p)$, $S(p)$ and Q be subgroups of nilpotent tensors $\eta(p)$ of height not greater than p, incompatibility tensors $\mathrm{s}(p)$ of height p, and compatible tensors $\mathbf{q}$ with zero divergence. We denote the remaining compatible tensors by ξ; they are characterized by the property $\operatorname{div} \xi \neq 0$ and form the subgroup Ξ (if they are supplemented by the zero tensor). Evidently $\mathrm{K} = S \oplus Q$, $\mathrm{H}(1) = Q \oplus \Xi$.

In application to the tensor $\eta(2)$ the representation (2.5) takes the form $\eta(2) = \mathrm{s}(2) + \mathbf{q} + \xi = \mathrm{s}(2) + \eta(1)$, and the direct sum

$$\mathrm{H}(2) = S(2) \oplus Q \oplus \Xi = S(2) \oplus \mathrm{H}(1) \tag{3.1}$$

corresponds to it because of the single-valuedness.

Lemma 3.1. $\operatorname{Ink} S(2) = Q$, $S(2) = \operatorname{Res} Q$.

Proof. The second relationship is obtained from the first if the operator Res works. To obtain the first relationship, we act on $S(2)$ with the operator Ink and on Q with the operator Res. From the definition of the height of an element we arrive at the inclusions $\mathrm{Ink}\, S(2) \subseteq Q$, $\mathrm{Res}\, Q \subseteq S(2)$, we act on the second with the operator Ink and compare with the first.

Lemma 3.2. There exists the isomorphism $S(2) \simeq Q$.

Proof. It follows from (3.1) and Lemma 3.1 that the operator Ink generates the homomorphism $\mathrm{H}(2) \to Q \subset \mathrm{H}(1)$, where the kernel of the homomorphism is $\mathrm{H}(1)$. Then by the theorem on homomorphisms, the isomorphism $Q \simeq \mathrm{H}(2)/\mathrm{H}(1)$ holds. On the other hand, the isomorphism $S(2) \simeq \mathrm{H}(2)/\mathrm{H}(1)$ follows from (3.1), which indeed proves the lemma.

Theorem 3.1. For any natural $p > 1$ there exists an isomorphism $S(p) \simeq Q$.

The assertion is proved by induction whose basis is Lemma 3.2. The method is analogous to the proof of Lemma 3.1.

The following generalization of Lemma 3.1 can be obtained as a corollary:

$$\mathrm{Ink}\, S(p) = S(p-1), \quad S(p) = \mathrm{Res}\, S(p-1), \; p > 2$$

Theorem 3.2. For every natural number $p > 1$ the following representation is valid

$$\mathrm{H}(p) = S(p) \oplus \mathrm{H}(p-1)$$

Successive application of this theorem permits obtaining the relationship

$$\mathrm{H}(p) = \bigoplus_{i=2}^{p} S(i) \oplus \mathrm{H}(1)$$

from which the following result

$$\mathrm{Ink}\,\mathrm{H}(p) = \bigoplus_{i=2}^{p-1} S(i) \oplus Q, \quad \mathrm{H}(p) = \mathrm{Ink}\,\mathrm{H}(p+1) \oplus \Xi$$

$$\mathrm{H}(p) = \mathrm{Res}\Big(\bigoplus_{i=2}^{p-1} S(i) \oplus Q\Big) \oplus \mathrm{H}(1)$$

Only the subgroups Ξ and $S(\infty)$ take part in the invariant decomposition proposed in the supplement to /2/. The subgroup Ξ consists of deformation tensors; they belong to the kernel of the operator Ink and are not images of any tensors since they cannot be represented in the form $\boldsymbol{\xi} = \mathrm{Ink}\,\boldsymbol{\eta}$. The subgroup $S(\infty)$ consists of non-nilpotent incompatibility tensors and is characterized by an automorphism generated by the operator Ink.

4. The height index will be omitted in the notation below, with the sole exception of the special case $p = 1$. We shall utilize the single-valued decomposition of the arbitrary tensor $\boldsymbol{\eta}$ into the deviatior $\boldsymbol{\eta}^*$ and spherical $\boldsymbol{\eta}^\circ$ parts

$$\boldsymbol{\eta} = \boldsymbol{\eta}^* + \boldsymbol{\eta}^\circ \tag{4.1}$$

We use the known identity /7/

$$\boldsymbol{\varkappa} = \nabla\nabla\cdot\cdot\boldsymbol{\eta}\boldsymbol{\delta} - \nabla\nabla\cdot\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla\nabla + \Delta\boldsymbol{\eta} + \nabla\nabla\,\mathrm{tr}\,\boldsymbol{\eta} - \Delta\,\mathrm{tr}\,\boldsymbol{\eta}\boldsymbol{\delta} \tag{4.2}$$

Here $\boldsymbol{\delta}$ is the unit tensor, tr and ∇ are operators of taking the trace of a tensor and of covariant differentiation. If $\boldsymbol{\eta} \equiv \mathbf{s} \in S$, then $\nabla\cdot\mathbf{s} = 0$ and (4.2) takes the form $\boldsymbol{\varkappa} = \Delta\mathbf{s} + \nabla\nabla\,\mathrm{tr}\,\mathbf{s} - \Delta\,\mathrm{tr}\,\mathbf{s}\boldsymbol{\delta}$. Expanding $\boldsymbol{\varkappa}$ and s into the sum (4.1) and using the notation $(\boldsymbol{\varkappa})^* \equiv \mathrm{Ink}\,\mathbf{s}^*$, $(\boldsymbol{\varkappa})^\circ \equiv \mathrm{Ink}\,\mathbf{s}^\circ$, we obtain from the latter relationships

$$(\boldsymbol{\varkappa})^* = \Delta\mathbf{s}^*, \quad (\boldsymbol{\varkappa})^\circ = \nabla\nabla\,\mathrm{tr}\,\mathbf{s}^\circ - \frac{2}{3}\Delta\,\mathrm{tr}\,\mathbf{s}^\circ\boldsymbol{\delta} \tag{4.3}$$

$$\boldsymbol{\varkappa}^* = \Delta\mathbf{s}^* + \nabla\nabla\,\mathrm{tr}\,\mathbf{s} - \frac{1}{3}\Delta\,\mathrm{tr}\,\mathbf{s}\boldsymbol{\delta}, \quad \boldsymbol{\varkappa}^\circ = -\frac{1}{3}\Delta\,\mathrm{tr}\,\mathbf{s}\boldsymbol{\delta} \tag{4.4}$$

In the general case, we obtain from (4.2)

$$\boldsymbol{\varkappa}^* = \frac{2}{3}\nabla\nabla\cdot\cdot\boldsymbol{\eta}\boldsymbol{\delta} - \nabla\nabla\cdot\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla\nabla + \Delta\boldsymbol{\eta}^* + \nabla\nabla\,\mathrm{tr}\,\boldsymbol{\eta} - \frac{1}{3}\Delta\,\mathrm{tr}\,\boldsymbol{\eta}\boldsymbol{\delta} \tag{4.5}$$

$$\boldsymbol{\varkappa}^\circ = \frac{1}{3}(\nabla\nabla\cdot\cdot\boldsymbol{\eta} - \Delta\,\mathrm{tr}\,\boldsymbol{\eta})\,\boldsymbol{\delta}$$

$$(\boldsymbol{\varkappa})^* = \nabla\nabla\cdot\cdot\boldsymbol{\eta}^*\boldsymbol{\delta} - \nabla\nabla\cdot\boldsymbol{\eta}^* - \boldsymbol{\eta}^*\cdot\nabla\nabla + \Delta\boldsymbol{\eta}^* \tag{4.6}$$

$$(\boldsymbol{\varkappa})^\circ = \frac{1}{3}(\nabla\nabla\,\mathrm{tr}\,\boldsymbol{\eta}^\circ - \Delta\,\mathrm{tr}\,\boldsymbol{\eta}^\circ\boldsymbol{\delta})$$

We examine the action of the operator Ink on the deviator and spherical incompatibility tensors. It follows directly from (4.4) that the operator Ink takes the incompatibility deviator $\mathbf{s}^*$ into the deviator $\boldsymbol{\varkappa}^*$. If $\mathbf{s} = \mathbf{s}^\circ$, then if follows from the condition $\nabla\cdot\mathbf{s} = 0$ that $\nabla\,\mathrm{tr}\,\mathbf{s}^\circ = 0$, and then we obtain $(\boldsymbol{\varkappa})^* = (\boldsymbol{\varkappa})^\circ = 0$ from (4.3), i.e. $\mathbf{s}^\circ \in Q$.

Therefore, no spherical incompatibility tensors exist; the deviator of the incompatibility tensor is always different from zero.

Let $s \in S$, we decompose it into the sum (4.1). Since $s^\circ \notin S$ then also $s^* \notin S$. Hence, either the incompatibility tensor is a deviatior or its deviatoric and spherical parts are not incompatibility tensors. It follows from (4.4) and (4.5) that in the general cases of $s \in S$ or $\eta \in H$ the operator Ink takes them over into $\varkappa \in K$, which are neither deviators nor spherical tensors.

It can be concluded on the basis of Lemma 1.1 that the operator Res restores the incompatibility deviator s^* from the deviator $\varkappa^*$. The subspace of incompatibility deviators is invariant for the operators Res and Ink. The deviator representation $\eta^* = s^* + \xi'^*$ is valid for an arbitrary deviatoric.

5. We decompose $\eta \in H$ into the sum (4.1) and we use the notation $(\varkappa)^* \equiv \mathrm{Ink}\,\eta^*$, $(\varkappa)^\circ \equiv \mathrm{Ink}\,\eta^\circ$. Because of the linearity of the operator Ink, we have $\varkappa = (\varkappa)^* + (\varkappa)^\circ$, and correspondingly

$$K = \mathrm{Ink}\,H,\ (K)^* = \mathrm{Ink}\,H^*,\ (K)^\circ = \mathrm{Ink}\ H^\circ,\ K = (K)^* + (K)^\circ \tag{5.1}$$

We define the subgroup E in $H(1)$ in the form of the direct sum

$$E = E^* \oplus E^\circ \tag{5.2}$$

where E^* and E° are subgroups of compatibile deviators e^* and spherical tensors e°. The subgroup E is a part of the kernel $H(1)$ of the operator Ink which is absolutely compatible in the sense that any element $e \in E$ has compatible and deviator and spherical parts. The subgroup E intersects Q and Ξ.

The subgroup E° is quite lean since it follows from (4.6) that $\nabla\nabla\,\mathrm{tr}\,e^\circ = 0$. As is known /8/, in a Cartesian system this is a linear function of the coordinates.

The single-valuedness of the decomposition (4.1) permits writing $H(1) = H^*(1) \oplus H^\circ(1)$. Since $E \subset H(1)$, $E^* \subset H^*(1)$, $E^\circ \subset H^\circ(1)$, then taking account of (5.2) we conclude that the decomposition

$$H(1)/E = H^*(1)/E^* \oplus H^\circ(1)/E^\circ \tag{5.3}$$

is valid for the factor-group H/E

We consider the subgroups $H^*(1)$ and $H^\circ(1)$. The deviators $\eta^*(1) \in H^*(1)$ can be compatible (belong to E^*) or incompatible. We form co-sets of incompatible deviators in the subgroup E^* and we take an arbitrary representative η_0^* from each class. The set obtained is isomorphic to the factor-group $H^*(1)/E^*$ and, therefore, itself forms a group $H_0^* \subset H^*$. We proceed analogously with the subgroup $H^\circ(1)$. Therefore, the isomorphisms $H^*(1)/E^* \simeq H_0^*$, $H^\circ(1)/E^\circ \simeq H_0^\circ$. Hence, by virtue of (5.3), a subgroup $H_0 \simeq H(1)/E$ exists in the subgroup $H(1)$. Correspondingly, the decompositions

$$\begin{gathered} H_0 = H_0^* \oplus H_0^\circ,\ H^*(1) = H_0^* \oplus E^*,\ H^\circ(1) = H_0^\circ \oplus E^\circ, \\ H(1) = H_0 \oplus E \end{gathered} \tag{5.4}$$

are valid.

It follows from the first relationship that every $\eta_0 \in H_0$, which is represented in conformity with (4.1) in the form $\eta_0 = \eta_0^* + \eta_0^\circ$, possesses the property

$$\mathrm{Ink}\ \eta_0^\circ = -\mathrm{Ink}\ \eta_0^* \tag{5.5}$$

The subgroup H_0° consists of incompatible spherical tensors η_0° for which incompatible deviators η_0^* corresponding to the condition (5.5) exist. The imbeddings $H_0^* \subset H^*$ and $H_0^\circ \subset H^\circ$ are obvious. Meanwhile, it follows from the Lemma 5.1 to be presented below that the elements η_0 are determined for any $\eta_0^\circ \in H^\circ$. This permits the conclusion that $H_0^\circ = H^\circ$, and correspondingly, the first relationship in (5.4) can be written in the form $H_0 = H_0^* \oplus H^\circ$. It hence follows that $(K)^\circ \subset (K)^*$.

Indeed, let us consider the action of the operator Ink on the subgroup in the last decomposition. Mapping by using Ink transfers H° into $(K)^\circ$ and H_0^* into a certain subgroup $(K)_0^*$, here $H_0 \subset H(1)$ goes over into zero. But this means that the subgroups $(K)^\circ$ and $(K)_0^*$ consist of mutually opposite elements, i.e., agree as sets and groups. Since $(K)_0^* \subset (K)^*$, then $(K)^\circ \subset (K)^*$. Hence, because of the last relationship in (5.1) we conclude that $K = (K)^*$. Therefore, there is proved the theorem.

Theorem 5.1. The set of tensor fields with zero divergence agrees with the set of images (by using Ink) of tensor-deviators.

For incompatibility tensors this result is formulated in the form $S = (S)^* \subset (K)^*$.

By condition (5.5) the element η_0 in a given spherical part η_0° is determined just to the accuracy of $e^* \in E^*$. Indeed, (5.5) means that $\eta_0 = \operatorname{def} u_0$ and giving the spherical part of the tensor is equivalent to the condition $\operatorname{div} u_0 = \operatorname{tr} \eta_0^\circ$ for the vector u_0. For a single-valued restoration of the vector u_0 it is necessary to give its rotation and boundary condition also, which is associated with the elements e^* and q^*. The potential part of the vector u_0 is related to the change in volume /9/, hence, for definiteness, we set $\operatorname{rot} u_0 = 0$. Any $\eta^\circ \in H^\circ$ can be the spherical part of η_0° since a particular solution of the system $\operatorname{div} u_0 = \operatorname{tr} \eta^\circ, \operatorname{rot} u_0 = 0$ exists for any piecewise-continuously differentiable right side. Therefore there is proved the lemma.

Lemma 5.1. For any piecewise-continuously differentiable spherical part $\eta^\circ \in H^\circ$ in the domain V the expression

$$\eta_0(x) = \nabla\nabla\varphi(x),\ \varphi(x) = \Pi_V(\rho \operatorname{tr} \eta^\circ)$$

single-valuedly defines the element $\eta_0 \in H_0 \subset H(1)$.

The vector $u_0 = \nabla\varphi$ is continuous and piecewise-continuously differentiable, meaning that it is also single-valued in the whole space of functions.

The following theorem illustrates Theorem 5.1 and can be of interest in connection with problems of the mechanics of incompressible media.

Theorem 5.2. Every piecewise continuously differentiable tensor η in the domain V can be decomposed into an incompatible deviator and a compatible part.

Proof. We define the element $\eta_0 \in H_0$ in the decomposition (4.1) by the spherical part η° such that $\eta_0 = \eta_0^* + \eta^\circ$, and expressing η° from this, substitute it into the decomposition (4.1)

$$\eta = (\eta^* - \eta_0^*) + \eta_0$$

This result was the basis for the method of equivalent modelling of stresses in an incompressible material due to given incompatible strains /10/.

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